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FOR

ENHANCING UTILITY AND DIVERSIFYING MODEL RISK IN A PORTFOLIO OPTIMIZATION FRAMEWORK

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ENHANCING UTILITY AND DIVERSIFYING MODEL RISK IN A PORTFOLIO OPTIMIZATION FRAMEWORK

[0001] This is a continuation-in-part of application serial No. 09/151,715, filed on September 11, 1998, that is currently pending.

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FIELD OF THE INVENTION

[0003] The invention relates generally to the field of financial advisory services. More particularly, the invention relates to a portfolio optimization process that diversifies model risk by favoring a more diversified portfolio over other portfolios with similar characteristics.

BACKGROUND OF THE INVENTION

[0004] From a set of N financial products (N > 1), an infinite number of portfolios are available for investment. Existing computer financial analysis systems (also referred to as "portfolio optimizers") purport to help individuals select portfolios to meet their needs. These systems typically implement mathematical models based upon standard optimization techniques involving mean-variance optimization theory. According to the mean-variance approach to portfolio selection, an optimal portfolio of financial products may be identified with reference to an investor's preference for various combinations of risk and return and the set of efficient portfolios (also referred to as the efficient set or the

efficient frontier). Figure 1 illustrates a feasible set of portfolios that represents all the portfolios that maybe formed from a particular set of financial products. The arc AC represents an efficient set of portfolios that each provide the highest expected return for a given level of risk. A portfolio's risk is typically measured by the standard deviation of returns. In general, there are many portfolios that have almost the same expected return and about the same level of risk as any efficient portfolio (e.g., portfolio B and portfolio E). Since statistical estimates of expected returns and risk are used to calculate efficient portfolios, the calculated efficient set could deviate from the true efficient set. When "model risk" is considered, portfolios in an error space surrounding an optimal portfolio are virtually indistinguishable. By "model risk," what is meant is the uncertainty/risk in the mathematical models employed and errors that may be introduced when estimating the properties of the financial products based upon historical data which may contain inaccuracies, such as statistical noise or measurement error, for example. An example of a problem induced by measurement error is the potential for highly concentrated estimated efficient portfolios. For instance, consider an asset that has a large positive error in its expected return estimate. Efficient portfolios constructed ignoring the possibility of this large positive error may yield portfolios with highly concentrated positions in this asset.

[0005] Existing portfolio optimizers typically ignore model risk, likely because of the great amount of processing that is thought to be required to identify and select from the many indistinguishable portfolios. Prior art portfolio optimizers are notorious for recommending portfolios that have counterintuitive properties, such as highly concentrated positions in individual assets or asset classes. For example, the typical portfolio optimizer, having ignored portfolio E because it is not in the efficient set, would suggest portfolio B which may include highly concentrated holdings in one of the underlying N assets. Such recommendations make users skeptical of the results of traditional portfolio optimizers and discourage adoption of such investment tools.

[0006] One way investment managers have traditionally attempted to compensate for the inadequacies of portfolio optimizers is by imposing constraints or bounds on the optimizer in one or more dimensions. For example, an investment manager may limit exposures to certain asset classes, limit short positions, etc. While these manual constraints can be implemented with knowledge of the bounded universe from which the portfolio will ultimately be built, they have several limitations. First, these manual techniques do not take the cost of imposing constraints on the optimization process into account. Additionally, manual solutions are typically only practical when the universe from which the portfolio can be drawn is limited to one set of mutual funds, asset classes, or financial products.

[0007] In view of the foregoing, what is needed is a generalized portfolio diversification approach that produces recommended portfolios that take into account inherent model risk and with which users will be intuitively comfortable, thereby fostering the adoption of optimization tools. Additionally, rather than arbitrarily spreading assets out, it is desirable for the decision to pursue more diversity in a portfolio to consider the cost of such diversity, in terms of its effect on expected return, risk, and/or utility, for example. Finally, it would be advantageous for the diversification approach to be broadly applicable to the universe of financial products.

SUMMARY OF THE INVENTION

[0008] A portfolio optimization process that diversifies model risk by favoring a more diversified portfolio over other portfolios with similar characteristics is described. Broadly stated the present invention involves determining an initial portfolio, performing diversification processing to identify one or more alternative portfolios having increased diversification, and selecting a recommended portfolio from the initial portfolio or the one or more alternative portfolios based upon a set of one or more criteria.

[0009] According to one aspect of the present invention, a more diverse portfolio may be selected over an initial portfolio in order to diversify model risk with reference to a predetermined diversity budget. An initial portfolio of financial products is determined from an available set of financial products. One or more dimensions of an error space are searched for an alternate portfolio that is more diverse than the initial portfolio. A cost associated with the alternate portfolio is then calculated by comparing the difference between a characteristic of the initial portfolio and a corresponding characteristic of the alternate portfolio. Finally, the alternate portfolio is selected as the recommended portfolio if the cost is less than or equal to the predetermined diversity budget.

[0010] According to another aspect of the present invention an intelligent search is performed for a diverse portfolio that meets a predetermined diversity budget. An initial portfolio is determined based upon an available set of financial products. The cost associated with more diversified portfolios compared to the initial portfolio is considered and one of the more diversified portfolios is selected that has an associated cost that is less than or equal to the predetermined diversity budget.

[0011] Other features of the present invention will be apparent from the accompanying drawings and from the detailed description which follows.

BRIEF DESCRIPTION OF THE DRAWINGS

[0012] The present invention is illustrated by way of example, and not by way of limitation, in the figures of the accompanying drawings and in which like reference numerals refer to similar elements and in which:

[0013] Figure 1 illustrates a feasible set of portfolios that can be formed from a set of financial products.

[0014] Figure 2 illustrates a financial advisory system according to one embodiment of the present invention.

[0015] Figure 3 is an example of a computer system upon which one embodiment of the present invention may be implemented.

[0016] Figure 4 is a simplified block diagram illustrating one embodiment of a financial analysis system that may employ the diversification mechanism of the present invention.

[0017] Figure 5 is a flow diagram illustrating portfolio optimization processing according to one embodiment of the present invention.

[0018] Figure 6 is a flow diagram illustrating diversification processing according to one embodiment of the present invention.

[0019] Figure 7 is a flow diagram illustrating diversification processing according to another embodiment of the present invention.

[0020] Figure 8 is a flow diagram illustrating the generation of a more diverse portfolio according to one embodiment of the present invention.

[0021] Figure 9A illustrates an initially identified optimal portfolio.

[0022] Figure 9B illustrates the effect of a maximum exposure constraint on the portfolio of Figure 9A.

[0023] Figure 9C illustrates a diversified portfolio after one or more stopping conditions have been achieved.

[0024] Figure 10 conceptually illustrates an approach for quickly finding a diversified portfolio that meets the diversity budget according to one embodiment of the present invention.

DETAILED DESCRIPTION

[0025] A mechanism is described for diversifying model risk. Such uncertainty/risk is inherent in the mathematical models and the historical data employed by portfolio optimizers, for example. The diversification mechanism described herein may efficiently search an error space proximate to an initially identified optimal portfolio for alternative portfolios that are more diverse than the initial portfolio and that are not too costly to implement in terms of differences in expected returns, risk and/or utility. According to embodiments of the present invention, after an initial efficient portfolio is identified by an optimization process, various characteristics of the initial portfolio may be used as a baseline by a diversification process to measure the cost of implementing more diverse portfolios having very similar expected return, risk, and/or utility characteristics as the initial portfolio. The more diverse portfolios may be located by searching various dimensions of an error space that is proximate to the initial portfolio. For example, the more diverse portfolios may be selected from a group of portfolios that have approximately the same level of risk and slightly lower expected returns than the initial portfolio or from a group of portfolios that have approximately the same expected returns but have a higher level of risk than the initial portfolio. In one embodiment, the diversification process favors more diverse portfolios over other portfolios with similar expected return characteristics by allocating a predetermined cost (referred to as the diversity budget) that can be spent in pursuit of diversity. In this manner, of the portfolios that are evaluated in a predefined error space, the most diverse portfolio that stays within the diversity budget will be selected. In other embodiments, other stopping conditions may also be employed to terminate the diversity processing. For example, the search for a more diverse portfolio than the current portfolio may stop when, among other things: (1) maintaining certain desirable characteristics of the initial portfolio constant is no longer feasible; (2) the number of financial products in the current portfolio exceeds a predetermined number of financial products; and/or (3) a certain number of iterations

have been performed and/or a certain number of alternate portfolios have been considered.

[0026] In the following description, for the purposes of explanation, numerous specific details are set forth in order to provide a thorough understanding of the present invention. It will be apparent, however, to one skilled in the art that the present invention may be practiced without some of these specific details. In other instances, well-known structures and devices are shown in block diagram form.

[0027] The present invention includes various steps, which will be described below. The steps of the present invention may be embodied in machine-executable instructions. The instructions can be used to cause a general-purpose or special-purpose processor which is programmed with the instructions to perform the steps of the present invention. Alternatively, the steps of the present invention may be performed by specific hardware components that contain hardwired logic for performing the steps, or by any combination of programmed computer components and custom hardware components.

[0028] The present invention may be provided as a computer program product which may include a machine-readable medium having stored thereon instructions which may be used to program a computer (or other electronic devices) to perform a process according to the present invention. The machine-readable medium may include, but is not limited to, floppy diskettes, optical disks, CD-ROMs, and magneto-optical disks, ROMs, RAMs, EPROMs, EEPROMs, magnetic or optical cards, or other type of media / machine-readable medium suitable for storing electronic instructions. Moreover, the present invention may also be downloaded as a computer program product, wherein the program may be transferred from a remote computer (e.g., a server) to a requesting computer (e.g., a client) by way of data signals embodied in a carrier wave or other propagation medium via a communication link (e.g., a modem or network connection).

[0029] While, embodiments of the present invention will be described with reference to a financial advisory system, the method and apparatus described herein are equally

applicable to other types of asset allocation applications, financial planning applications, investment advisory services, and financial product selection services, automated financial product screening tools such as electronic personal shopping agents and the like.

SYSTEM OVERVIEW

[0030] The present invention may be included within a client-server based financial advisory system 200 such as that illustrated in Figure 2. According to the embodiment depicted in Figure 2, the financial advisory system 200 includes a financial staging server 220, a broadcast server 215, a content server 217, an AdviceServer™ 210 (AdviceServer is a trademark of Financial Engines, Inc., the assignee of the present invention), and a client 205.

[0031] The financial staging server 220 may serve as a primary staging and validation area for the publication of financial content. In this manner, the financial staging server 220 acts as a data warehouse. Raw source data, typically time series data, may be refined and processed into analytically useful data on the financial staging server 220. On a monthly basis, or whatever the batch processing interval may be, the financial staging server 220 converts raw time series data obtained from data vendors from the specific vendor's format into a standard format that can be used throughout the financial advisory system 200. Various financial engines may also be run to generate data for validation and quality assurance of the data received from the vendors. Any calibrations of the analytic data needed by the financial engines may be performed prior to publishing the final analytic data to the broadcast server 215.

[0032] The broadcast server 215 is a database server. As such, it runs an instance of a Relational Database Management System (RDBMS), such as Microsoft™ SQL-Server, Oracle™ or the like. The broadcast server 215 provides a single point of access to all fund information and analytic data. When advice servers such as AdviceServer 210 need data, they may query information from the broadcast server database. The broadcast

server 215 may also populate content servers, such as content server 217, so remote implementations of the AdviceServer 210 need not communicate directly with the broadcast server 215. The AdviceServer 210 is the primary provider of services for the client 205. The AdviceServer 210 also acts as a proxy between external systems, such as external system 225, and the broadcast server 215 or the content server 217.

[0033] According to the embodiment depicted, the user may interact with and receive feedback from the financial advisory system 200 using client software which may be running within a browser application or as a standalone desktop application on the user's personal computer 205. The client software communicates with the AdviceServer 210 which acts as a HTTP server.

AN EXEMPLARY COMPUTER SYSTEM

[0034] Having briefly described an exemplary financial advisory system 200 which may employ various features of the present invention, a computer system 300 representing an exemplary client 105 or server in which features of the present invention may be implemented will now be described with reference to Figure 3. Computer system 300 comprises a bus or other communication means 301 for communicating information, and a processing means such as processor 302 coupled with bus 301 for processing information. Computer system 300 further comprises a random access memory (RAM) or other dynamic storage device 304 (referred to as main memory), coupled to bus 301 for storing information and instructions to be executed by processor 302. Main memory 304 also may be used for storing temporary variables or other intermediate information during execution of instructions by processor 302. Computer system 300 also comprises a read only memory (ROM) and/or other static storage device 306 coupled to bus 301 for storing static information and instructions for processor 302.

[0035] A data storage device 307 such as a magnetic disk or optical disc and its corresponding drive may also be coupled to computer system 300 for storing information

and instructions. Computer system 300 can also be coupled via bus 301 to a display device 321, such as a cathode ray tube (CRT) or Liquid Crystal Display (LCD), for displaying information to a computer user. For example, graphical depictions of expected portfolio performance, asset allocation for an optimal portfolio, charts indicating short- and long-term financial risk, icons indicative of the probability of achieving various financial goals, and other data types may be presented to the user on the display device 321.. Typically, an alphanumeric input device 322, including alphanumeric and other keys, is coupled to bus 301 for communicating information and/or command selections to processor 302. Another type of user input device is cursor control 323, such as a mouse, a trackball, or cursor direction keys for communicating direction information and command selections to processor 302 and for controlling cursor movement on display 321.

[0036] A communication device 325 is also coupled to bus 301 for accessing remote servers, such as the AdviceServer 210, or other servers via the Internet, for example. The communication device 325 may include a modem, a network interface card, or other well known interface devices, such as those used for coupling to Ethernet, token ring, or other types of networks. In any event, in this manner, the computer system 300 may be coupled to a number of servers via a conventional network infrastructure, such as a company's Intranet and/or the Internet, for example.

EXEMPLARY FINANCIAL ANALYSIS SYSTEM

[0037] Figure 4 is a simplified block diagram illustrating a financial analysis system 400 in which one embodiment of the present invention may be used. Generally, the financial advisory system 400 includes a simulation module 440, a portfolio optimization module 456, and a user interface (UI) 460. The UI 460 may include various mechanisms for data input and output to provide the user with a means of interacting with and receiving feedback from the financial advisory system 400, respectively. Both the simulation module 440 and the portfolio optimization module may receive input data

from the user interface (UI) 460 and provide data, such as financial products' exposures to various factors, probability distributions, and recommended portfolios of financial products, to the UI 460.

[0038] The simulation module 440 may include a simulation engine for empirically generating draws from a random distribution. According to the embodiment depicted, the simulation module 440 further includes a pricing module 410, a factor module 420, and a style analysis module 430.

[0039] The pricing module 410 may generate pricing data for one or more assets. In one embodiment, pricing module 410 generates pricing data for three assets (e.g., short-term bonds, long-term bonds and U.S. equities). These assets are used as core assets by simulation module 440 for simulation functions. Alternatively, the core assets may be different types of assets, such as U.S. equities and bonds (making no distinction between short-term and long-term bonds). Of course, a different number of core assets may also be used.

[0040] In one embodiment, pricing module 410 generates a number of asset scenarios. Each scenario is an equally likely outcome based on the inputs to financial advisory system 400. By generating a number of scenarios with pricing module 410, financial advisory system 400 may generate statistics for different projected asset valuations. For example, financial advisory system 400 may provide probability distributions for each projected asset valuation.

[0041] Factor module 420 receives core asset pricing data from pricing module 410 and maps the data onto a set of factors. Factors output by factor module 420 are used by returns-based style analysis module 430 to generate style exposures for particular assets. Factor modules and style analysis are well known in the art and are not described in greater detail herein. Factor module 420 and style analysis module 430 may perform the functions as described in "Asset allocation: Management style and performance

measurement," by William F. Sharpe, Journal of Portfolio Management, Vol. 18, No. 2, which is hereby incorporated by reference.

[0042] The portfolio optimization module 456 may determine optimal portfolios based on input provided to financial advisory system 400 via UI 460. In the embodiment depicted, the portfolio optimization module 456 further comprises a diversification module 455 and an optimization module 450. The optimization module 450 calculates the utility maximizing set of financial products under a set of constraints defined by the user and the available feasible investment set. In one embodiment, the calculation is based upon a mean-variance optimization of the financial products.

[0043] The diversification module 455 manages diversification processing and evaluates the cost of performing diversification. As will be described further below, during diversification processing, the diversification module 455 may cause the optimization module 450 to perform several iterations of optimization processing with various constraints, such as a maximum exposure to any individual financial product and/or a minimum exposure to any individual financial product. In one embodiment, the diversity budget is set to an appropriate default level. The appropriate default level may be determined by tuning a parameter utilized the financial analysis system until satisfactory results are achieved, for example. In another embodiment, the user may provide a preference for diversification via the UI 460, which may in turn be used to determine the diversity budget. Depending upon the user's expressed preference for diversity, a diversity budget, typically from 0 basis points to 16 basis points may be allocated, for example, corresponding to a preference for no diversity and a high preference for diversity, respectively. Importantly, as will be discussed further below, rather than arbitrarily spreading assets out, the decision to pursue more diversity in a portfolio by the diversification module 455 is made after explicitly considering cost of such diversity, in terms of its effect on expected return, risk, and/or utility, for example.

[0044] Importantly, the portfolio optimization module 456 may execute on a server or on the same computer upon which the UI 460 resides.

[0045] Further description of a financial advisory system that may incorporate various features of the present invention is disclosed in a copending U.S. Patent application entitled "USER INTERFACE FOR A FINANCIAL ADVISORY SYSTEM," Application No. 09/904,707, filed on July 12, 2001 that is assigned to the assignee of the present invention and which is hereby incorporated by reference.

PORTFOLIO OPTIMIZATION

[0046] In general, portfolio optimization is the process of determining a set of financial products that maximizes the utility function of a user. According to one embodiment, portfolio optimization processing assumes that users have a mean-variant utility function, namely, that people like having more expected wealth and dislike volatility of wealth. Based on this assumption, given a user's risk tolerance, the portfolio optimization module 456 may calculate an initial mean-variance efficient optimal portfolio from a set of financial products that are available to the user. Depending upon the user's diversity preference, other more diversified portfolios may then be considered for purposes of diversifying model risk. Preferably, both the optimization problem and the diversification problem are expressed as a series of one or more Quadratic Programming (QP) problems. QP is a technique for solving optimization problems involving quadratic (squared terms) objective functions with linear equality and/or inequality constraints. A number of different QP techniques exist, each with different properties. For example, some are better for suited for small problems, while others are better suited for large problems. Some are better for problems with very few constraints and some are better for problems with a large number of constraints. According to one embodiment of the present invention, when QP is called for, an approach referred to as an "active set" method is employed herein. The active set method is explained in Gill, Murray, and

Wright, "Practical Optimization," Academic Press, 1981, Chapter 5. Advantageously, if the diversification problem can be structured as a series of one or more QP problems, then interactive applications, such as software that provides financial advice to individuals, may perform diversification processing in real-time.

[0047] Referring now to Figure 5, portfolio optimization processing according to one embodiment of the present invention will now be described. In one embodiment, the steps described below may be performed under the control of a programmed processor, such as processor 302 resident in client 205, or one of the servers 220, 215, 217, or 210. At step 510, an initial optimal portfolio is determined. According to one embodiment of the present invention, the optimal portfolio is a mean-variance efficient portfolio which may be determined with reference to user-supplied data regarding his/her desirability for various combinations of risk and return. In this example, wealth in real dollars may be optimized by maximizing the following mean-variance utility function by determining portfolio proportions (X_i):

$$U = E(W) - \frac{Var(W)}{\tau} \qquad \text{(EQ \#1)}$$

where for a given scenario,

$E(W)$ is the expected value of wealth

$Var(W)$ is the variance of wealth

τ is the user's risk tolerance

$$W = W_0 \sum_{i=1}^{n} X_i(1 + R_i) \qquad \text{(EQ \#2)}$$

where,

W_0 = initial wealth

R_i = random return on financial product i

X_i represents the recommended constant proportion of each contribution allocated to financial product i.

$0 \leq X_i \leq UB$

UB = Upper bound on maximum exposure

n is the number of financial products that are available for optimization.

Alternately, the following equation could be substituted for EQ #2:

$$W_T = X_1 \sum_{t=0}^{T-1} C_t \prod_{j=t+1}^{T} (1 + R_{j1}) + \ldots + X_n \sum_{t=0}^{T-1} C_t \prod_{j=t+1}^{T} (1 + R_{jn}) + g \qquad \text{(EQ \#2A)}$$

where,

X_i represents the recommended constant proportion of each net contribution that should be allocated to financial product i.

C_t represents the net contribution at time t,

R_{ji} represents the expected returns for financial product i in year j,

n is the number of financial products that are available for optimization,

g is the value of constrained assets for a given scenario,

The product of gross returns represents the compounding of values from year 1 to the horizon. Initial wealth in the portfolio is represented by contribution C_0.

[0048] Importantly, the financial product returns need not represent fixed allocations of a single financial product. Within the context of the optimization problem, any individual asset return may be composed of a static or dynamic strategy involving one or more financial products. For example, one of the assets may itself represent a constant re-balanced strategy over a group of financial products. Moreover, any dynamic strategy that can be formulated as an algorithm may be incorporated into the portfolio optimization. For example, an algorithm which specifies risk tolerance which decreases with the age of the user could be implemented. It is also possible to incorporate path dependent algorithms (e.g., portfolio insurance).

[0049] According to Equation #2A, contributions are made from the current year to the year prior to retirement. Typically, a contribution made at time t will be invested from time t until retirement. An exception to this would be if a user specifies a withdrawal, in which case a portion of the contribution may only be held until the expected withdrawal date.

[0050] An alternative to the buy and hold investment strategy assumed above would be to implement a "constant mix" investment strategy or re-balancing strategy. For purposes of this example, it is assumed that the recommended fixed target asset-mix will be held in an account for each year in the future. Therefore, each year, assets will be bought and/or sold to achieve the target. Let f_i be the fraction of account wealth targeted for the i-th asset, then the sum of the fractions must equal one.

[0051] In the following "evolution" equations, nominal wealth aggregation is modeled for a single taxable account from the current time t = 0 to the time horizon t = T. It is assumed that "N" assets are in the account, labeled by the set of subscripts {i=1, ..., N}. The superscripts minus and plus are used to distinguish between the values of a variable just before, and just after, "settlement". The settlement "event" includes paying taxes on distributions and capital gains, investing new contributions, buying and selling assets to achieve the constant mix, and paying load fees. For example, $W^+(t)$ is the total wealth invested in all assets just after settlement at time "t". The evolution equations for the pre- and post-settlement values, the "dollars" actually invested in each asset, are:

$$(19a)\quad W_i^-(t) = \begin{cases} W_i^-(0), & t = 0, \\ [1 + R_i(t)] \cdot W_i^+(t-1) - \|k_i(t)\|, & 0 < t \le T, \end{cases}$$

$$(19b)\quad W_i^+(t) = \begin{cases} f_i \cdot W^+(t), & 0 \le t < T, \\ 0, & t = T. \end{cases}$$

[0052] In the above equation, the double-bar operator || || is equal to either its argument or zero, whichever is greater. From Eq.(19a), we see that the pre-settlement value at any time (after the initial time) is just the gross return on the post-settlement value of the previous time less the "positive-part" of any distribution, i.e. the "dividend". Here, $k_i(t)$ is the portion of the return of the i-th asset that is distributed, and $R_i(t)$ is the total nominal return on the i-th asset in the one-year period [t-1, t]. We also assume that an initial, pre-

settlement value is given for each asset. Eq.(19b) defines the post-settlement value in terms of the asset's constant mix and the total account value after settlement. Since we "cash-out" the portfolio at the time horizon, the final amount in each asset at t = T is zero. The pre- and post-settlement, total values are governed by the pair of equations:

$$(19c) \quad W^{-}(t) = \sum_{i=1}^{N} W_i^{-}(t), \quad 0 \le t \le T,$$
$$(19d) \quad W^{+}(t) = W^{-}(t) + C(t) + D(t) - L(t) - S(t), \quad 0 \le t \le T.$$

[0053] In Eq.(19d), C(t) is the nominal contribution to the account at time "t", D(t) is the total of all distributed "dividends", L(t) is the "leakage", the total amount paid in loads to both rebalance and to invest additional contributions, and S(t) is the "shrinkage", the total amount paid in taxes on distributions and capital gains. We note that $W^+(T)$ is the final horizon wealth after all taxes have been paid. The value of D(t), the total of all distributed dividends, is the sum of the positive distributions:

$$(19e) \quad D(t) = \sum_{i=1}^{N} \|k_i(t)\|, \quad 0 \le t \le T.$$

[0054] Similarly, the "leakage" L(t) is the total amount of dollars paid in loads, and $L_i(t)$ is the number of dollars paid in loads on just the i-th asset. These individual loads depend on l_i, the front-end load fee (a rate) on the i-th asset.

$$(19f) \quad L_i(t) = [l_i/(1-l_i)] \cdot \left\| W_i^{+}(t) - \|k_i(t)\| - W_i^{-}(t) \right\|, \quad 0 \le t \le T.$$
$$(19g) \quad L(t) = \sum_{i=1}^{N} L_i(t), \quad 0 \le t \le T.$$

[0055] If there is a short-term loss (negative distribution), the load fee paid on an asset's purchase is just a fixed fraction of the purchase price.[i] When there is a short-term gain (positive distribution), we can re-invest any part of it without load fees, and pay fees only on purchases in excess of the gain. Note that at the horizon, we "cash-out", and don't pay any load fees.

[0056] The equation for the "shrinkage" S(t), the total amount paid in taxes, has two terms. The first term is the tax on distributions and is multiplied by the marginal tax-rate; the second term is the tax on capital gains and is multiplied by the capital gains tax-rate.

$$(19h)\quad S(t)=\tau_m \cdot \sum_{i=1}^{N} k_i(t)+\tau_{cg}\cdot \sum_{i=1}^{N}[1-B_i(t-1)/W_i^-(t)]\cdot \left\|W_i^-(t)-W_i^+(t)\right\|,\quad 0\le t\le T.$$

[0057] In Eq.(19h), the capital gains tax depends on the basis $B_i(t)$, the total of all after-tax nominal-dollars that have been invested in the i-th asset up to time "t". Note that there can be either a capital gain or loss. The double-bar operator ensures that capital gains are triggered only when there is a sale of assets. At the horizon, we sell all assets, and automatically pay all taxes. The basis $B_i(t)$, evolves according to the following recursion equation:

$$(19i)\quad B_i(t)=\begin{cases} B_i(0), & t=0,\\ B_i(t-1)+\left\|W_i^+(t)-W_i^-(t)\right\|+L_i(t) & \\ -[B_i(t-1)/W_i^-(t)]\cdot\left\|W_i^-(t)-W_i^+(t)\right\|, & 0<t\le T. \end{cases}$$

[0058] Note that all new purchases are made with after-tax dollars, and add to the basis; all sales decrease the basis. Further, any load paid to purchase an asset adds to the basis. We assume that the initial basis $B_i(0)$ of an asset is either given, or defaults to the initial, pre-settlement value so that the average basis is initially equal to one.

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[i] The dollar amount of a load fee is proportional to the ratio l / (1- l). That's because our wealth variables are all measured as "net" loads. To see this, suppose we make a contribution c. After loads, we are left with W = (1- l) c. In terms of W, the amount we paid in loads is L = l c = [l / (1- l)] W.

[0059] A "constitutive" equation for $k_i(t)$ is needed to complete our system of equations. Short-term distributions depend on the "type" of asset; here we model mutual funds:

$$(20a) \quad k_i(t) = \begin{cases} k_i(0), & t = 0, \\ \kappa_i \cdot R_i(t) \cdot W_i^{+}(t-1), & 0 < t \le T. \end{cases}$$

[0060] Often, we set the initial distribution to zero, and assume that the asset's initial pre-settlement value has already accounted for any non-zero, initial value. We note that the distribution is proportional to the amount of wealth at "stake" during the prior-period. For mutual funds, we assume that the distribution is a fraction κ_i of the prior-period's total return, and therefore is also proportional to $R_i(t)$. Note that the distribution in Eq.(20a) can be a gain (positive) or a loss (negative). In contrast, the constitutive equation for stocks takes the form:

$$(20b) \quad k_i(t) = \begin{cases} k_i(0), & t = 0, \\ \kappa_i \cdot [1 + R_i(t)] \cdot W_i^{+}(t-1), & 0 < t \le T. \end{cases}$$

[0061] For stocks, the proportionality constant κ_i models a constant dividend "yield", and the distribution is always a gain (non-negative). For stocks (mutual funds), the distribution is proportional to the gross (simple) return.

[0062] Before we leave this section, a word on 401(k) plans and IRA's (with no load funds). For these accounts, the loads and taxes are ignored, and there is no basis in the asset. At "settlement", the user just re-balances their account. The evolution equations for these accounts is trivial in comparison to the equations for a general taxable account:

$$(21a)\quad W_i^+(t) = f_i \cdot W^+(t), \quad 0 \le t \le T,$$

$$(21b)\quad W^+(t) = \begin{cases} W^+(0), & t = 0, \\ \left(1 + \sum_{i=1}^{N} f_i \cdot R_i(t)\right) \cdot W^+(t-1) + C(t), & 0 < t \le T. \end{cases}$$

[0063] At the time horizon T, the total wealth in a non-taxable account is just $W^+(T)$. This is a pre-withdrawal total value. When retirement withdrawals are made from a tax-free account, they are taxed at the client's average tax-rate, τ_a. Therefore, the "after-tax" equivalent value is equal to "pre-tax" wealth $W^+(T)$ times the tax factor $(1 - \tau_a)$.

[0064] How do we aggregate taxable and non-taxable accounts to get total portfolio wealth? We choose non-taxable accounts as a baseline. If all the funds in a non-taxable account were converted to an annuity, and the annuity payments were taken as withdrawals, then the withdrawals would mimic a salary subject to income taxes. This is precisely the client's pre-retirement situation. Before aggregating a taxable account, we scale its "after-tax" value to this baseline using the formula:

$$(22)\quad W_{baseline} = W_{after-tax} / (1 - \tau_a).$$

[0065] Essentially, the baseline equivalent is obtained by grossing up values using the average tax-rate.

[0066] The evolution equation variables appear "implicitly" in the recursion relations. Hence, we need to "iterate" at each time step to solve for "explicit" variable values.[ii] We illustrate this process with an example. Consider the simple case where there are no distributions, contributions, or taxes; just loads, and a constant-mix strategy. Here, the evolution equations simplify to a single equation for the total, after-settlement wealth $W^+(t)$:

$$(23)\; W^+(t) = W^+(t-1)\cdot\sum_{i=1}^{N} f_i\cdot\left[1+R_i(t)\right] - \sum_{i=1}^{N} f_i\cdot\left[l_i/(1-l_i)\right]\cdot\left\| W^+(t) - \left[1+R_i(t)\right]\cdot W^+(t-1)\right\|.$$

Note, we only know $W^+(t)$ as an implicit function of $W^+(t-1)$, but given a guess for it's value, we can refine the guess by substituting it into the right-side of Eq.(23).

[0067] It's instructive to re-write Eq.(23) as the pair of equations in terms of an "effective" return $R_e(t)$:

$$(24a)\quad W^+(t) = [1+R_e(t)]\cdot W^+(t-1),$$
$$(24b)\quad R_e(t) = \sum_{i=1}^{N} f_i\cdot R_i(t) - \sum_{i=1}^{N} f_i\cdot[l_i/(1-l_i)]\cdot\left\| R_e(t) - R_i(t)\right\|.$$

[0068] Eq.(24a) is the evolution equation for a single asset with the effective return. Eq.(24b) is an implicit equation for the effective return $R_e(t)$ in terms of the asset returns $R_i(t)$. We solve for the effective return using iteration. When the loads are equal to zero, as expected, the effective return is just a weighted-average of the asset returns. Even when the loads are not zero, this average return is a good initial guess for the iteration

[ii] In practice a robust root-finding algorithm may be used rather than iteration.

procedure. In fact, using the average return as the initial guess and iterating once yields the following explicit approximation for the effective return:

$$(25a)\quad R_{wgt}(t) = \sum_{i=1}^{N} f_i \cdot R_i(t),$$

$$(25b)\quad R_e(t) \approx R_{wgt}(t) - \sum_{i=1}^{N} f_i \cdot l_i \cdot \left\| R_{wgt}(t) - R_i(t) \right\|.$$

Eq.(25b) is consistent with our intuition, and agrees well with higher order iterates.

[0069] To determine the mutual fund input moments we must first calculate the kernel moments. This procedure calculates successive annual kernel moments and averages the result. The resulting mean and covariance matrix is then utilized by the reverse optimization procedure and also as an input into the optimization procedure.

[0070] To calculate analytic core moments, first we must describe the wealth for each core asset for an arbitrary holding period. For each of the core assets, the resulting wealth from an arbitrary investment horizon can be written as: [Note, this is an approximation for equities]

$$W_{t,T} = \exp\left\{ \sum_{j=t}^{T-1} a + bX_{j+1} + c\Pi_{j+1} + d\delta_{j+1} + eX_j + f\Pi_j + g\delta_j \right\}$$

Where:

a, b, c, d, e, f, g	=	Constants
X_j	=	Real rate in year j
Π_j	=	Inflation rate in year j
δ_j	=	Dividend growth rate in year j

The expectation of wealth for any of the core assets given information at time zero is then:

$$E_0 W_{t,T} = e^{a(T-t)} E_0 e^{\sum_{j=t}^{T-1} eX_j + bX_{j+1}} E_0 e^{\sum_{j=t}^{T-1} f\Pi_j + c\Pi_{j+1}} E_0 e^{\sum_{j=t}^{T-1} g\delta_j + d\delta_{j+1}}$$

[0071] Since X, Π, and δ are independent, we can deal with each of these expectations separately. For example, consider the contribution in the above equation from inflation. The summation can be rewritten as:

$$E_0 \exp\left\{ \sum_{j=t}^{T-1} f\Pi_j + c\Pi_{j+1} \right\} = E_0 \exp\left\{ f\Pi_t + \left(\sum_{j=t+1}^{T-1} (f+c)\Pi_j \right) + c\Pi_T \right\}$$

[0072] Next, we need to use iterated expectations to determine this expectation. We can write the expectation at time zero as the repeated expectation over the various innovations. For example, the equation for inflation can be rewritten as:

$$\begin{aligned} E_0 \exp & \left\{ f\Pi_t + \left(\sum_{j=t+1}^{T-1} (f+c)\Pi_j \right) + c\Pi_T \right\} \\ &= E_{\varepsilon_1} E_{\varepsilon_2} \cdots E_{\varepsilon_T} \exp\left\{ f\Pi_t + \left(\sum_{j=t+1}^{T-1} (f+c)\Pi_j \right) + c\Pi_T \right\} \\ &= E_{\varepsilon_1} E_{\varepsilon_2} \cdots E_{\varepsilon_{T-1}} \exp\left\{ f\Pi_t + \left(\sum_{j=t+1}^{T-1} (f+c)\Pi_j \right) \right\} E_{\varepsilon_T}\left[e^{c\Pi_T} \right] \end{aligned}$$

Assuming inflation follows a modified square root process:

$$\Pi_t = \mu_\pi + \rho_\pi \Pi_{t-1} + \sigma_\pi \sqrt{\|\Pi_{t-1}\|}\varepsilon_t$$

Where || || denotes the Heaviside function

$$\| \Pi_t \| \equiv \begin{cases} 0 & \text{if} \quad \Pi_t \le 0 \\ \Pi_t & \text{if} \quad \Pi_t > 0 \end{cases}$$

[0073] Now we recursively start taking the expectations over epsilon starting at the end and working backward. So:

$$\begin{aligned} E_{\varepsilon_T}\left[e^{c\Pi_T}\right] &= E_{\varepsilon_T}\left[e^{c\mu_\pi + c\rho_\pi \Pi_{T-1} + c\sigma_\pi \sqrt{\|\Pi_{T-1}\|}\,\varepsilon_T}\right] \\ &\approx e^{c(\mu_\pi + \rho_\pi \Pi_{T-1} + \frac{1}{2} c\sigma_\pi^2 \Pi_{T-1})} \end{aligned}$$

Where the approximation is due to the Heaviside function.

[0074] Combining this with the above equation yields:

$$\begin{aligned} & E_{\varepsilon_1} E_{\varepsilon_2} \cdots E_{\varepsilon_{T-1}} \exp\left\{ f\Pi_t + \left(\sum_{j=t+1}^{T-1} (f+c)\Pi_j \right) \right\} E_{\varepsilon_T}\left[e^{c\Pi_t}\right] \\ &= E_{\varepsilon_1} E_{\varepsilon_2} \cdots E_{\varepsilon_{T-2}} \exp\left\{ f\Pi_t + \left(\sum_{j=t+1}^{T-1} (f+c)\Pi_j \right) \right\} E_{\varepsilon_{T-1}}\left[e^{c\mu_\pi + \left(c\rho_\pi + \frac{1}{2} c^2 \sigma_\pi^2 + c + f\right)\Pi_{T-1})} \right] \end{aligned}$$

[0075] In general for any time period t, an exponential linear function of Π has the following expectation:

$$
\begin{aligned}
E_{\varepsilon_t}\left[e^{A_J + B_J \Pi_t}\right] &= E_{\varepsilon_t}\left[e^{A_J + B_J\left(\mu_\pi + \rho_\pi \Pi_{t-1} + \sigma_\pi \|\Pi_{t-1}\| \varepsilon_t\right)}\right] \\
&= e^{A_J + B_J \mu_\pi + B_J \Pi_{t-1}\left(\rho_\pi + \frac{1}{2}\sigma_\pi^2 B_J\right)} \\
&= e^{A_J + B_J \mu_\pi + \left(B_J\left(\rho_\pi + \frac{1}{2}\sigma_\pi^2 B_J\right)\right)\Pi_{t-1}} \\
&= e^{A_{J-1} + B_{J-1}\Pi_{t-1}}
\end{aligned}
$$

[0076] The critical feature is that an exponential linear function of Π remains exponential linear after taking the expectation. This invariance allows for the backward recursion calculation. Only the constant (A) and the slope (B) are changing with repeated application of the expectation operator. The evolution of A and B can be summarized as

$$
\begin{aligned}
A_J &= A_{J+1} + \mu_\pi B_{J+1} \\
B_J &= B_{J+1}\left[\rho_\pi + \tfrac{1}{2}\sigma_\pi^2 B_{J+1}\right]
\end{aligned}
$$

In addition, the B_J coefficient has to be increased by (c + f) to account for the additional

Π_J term in the summation. To implement this recursive algorithm to solve for expected wealth, first define the following indicator variable:

$$I(t_1, t_2) = \begin{Bmatrix} 1 & \text{if } t_1 \le j \le t_2 \\ 0 & \text{Otherwise} \end{Bmatrix}$$

[0077] Next, the following algorithm may be employed:

InitialConditions $J = T,\ A_T = 0,\ B_T = c$

(1) $J = J - 1$

(2) $A_J = A_{J+1} + \mu_\pi B_{J+1}$

$B_J = B_{J+1}\left[\rho_\pi + \tfrac{1}{2}\sigma_\pi^2 B_{J+1}\right] + c \cdot I(t+1,\ T-1) + f \cdot I(t,\ T-1)$

(3) if $J = 0$, End

$E(W_{t,T}) = e^{A_1 + B_1 \Pi_0}$

(4) Go To (1)

[0078] The same technique applies to X since it is also a square root process. A similar technique can be used to create a recursive algorithm for the δ component. The only difference is that δ is an AR(1) process instead of a square root process.

In particular,

$$\delta_t = \mu_\delta + \rho_\delta\, \delta_{t-1} + \sigma_\delta\, \varepsilon_t$$

For this AR(1) process, the expectation is of the following form.

$$
\begin{aligned}
E_{\varepsilon_t}\left[e^{A_J + B_J \delta_t}\right] &= E_{\varepsilon_t}\left[e^{A_J + B_J(\mu_\delta + \rho_\delta \delta_{t-1} + \sigma_\delta \varepsilon_t)}\right] \\
&= e^{A_J + B_J \mu_\delta + \frac{1}{2}\sigma_\pi^2 B_J + B_J \rho_\delta \delta_{t-1}} \\
&= e^{A_{J-1} + B_{J-1}\delta_{t-1}}
\end{aligned}
$$

The evolution of A and B is thus summarized as:

$$
\begin{aligned}
A_J &= A_{J+1} + B_{J+1}\left(\mu_\delta + \tfrac{1}{2}\sigma_\delta^2\right) \\
B_J &= B_{J+1}\,\rho_\delta
\end{aligned}
$$

The recursive relationship for δ is then:

$$\text{InitialConditions} \quad J = T, \ A_T = 0, \ B_T = d$$

(1) $J = J - 1$

(2) $A_J = A_{J+1} + B_{J+1}\left(\mu_\delta + \tfrac{1}{2}\sigma_\delta^2\right)$

$B_J = B_{J+1}\rho_\delta + d \cdot I(t+1, T-1) + g \cdot I(t, T-1)$

(3) if $J = 0$, End

$E(W_{t,T}) = e^{A_1 + B_1\delta_0}$

(4) Go To (1)

[0079] This framework for calculating expected wealth can also be used to calculate the variance of wealth for an arbitrary holding period . From the definition of variance, we have:

$$V_0(W_{t,T}) = E_0(W_{t,T}^2) - E_0(W_{t,T})^2$$

but

$$W_{t,T}^{2} = \left[\exp\left\{\sum_{j=t}^{T-1} a + bX_{j+1} + c\Pi_{j+1} + d\delta_{j+1} + eX_{j} + f\Pi_{j} + g\delta_{j}\right\}\right]^{2}$$
$$= \exp\left\{\sum_{j=t}^{T-1} 2\left(a + bX_{j+1} + c\Pi_{j+1} + d\delta_{j+1} + eX_{j} + f\Pi_{j} + g\delta_{j}\right)\right\}$$

[0080] So the same technique can be used with a simple redefinition of the constants to be twice their original values. Similarly, the covariance between any two core assets can be calculated by simply adding corresponding constants and repeating the same technique.

[0081] For the current parameter values, the constants for Bills, Bonds, and Equities are:

	a	b	c	d	e	F	g
Bills	0.0077	0	-1	0	1	0.7731	0
Bonds	0.0642	-2.5725	-3.8523	0	2.5846	2.9031	0
Equities	0.0331	-2.4062	-3.7069	4.4431	2.48	2.79	-3.5487

[0082] Above, a methodology was described for calculating core asset analytic moments for arbitrary horizons. This section describes how these moments are translated into annualized moments. The procedure described in this section essentially calculates successive annual moments for a twenty (20) year horizon and computes the arithmetic average of these moments. These 'effective' annual moments may then be used as inputs into the reverse optimization procedure and the individual optimization problem.

[0083] For this calculation, first make the following definitions:

$$\mathrm{M}_t^j = \text{Expected return for } j^{\text{th}} \text{ asset over the period } t, t+1$$

$$\mathrm{Cov}_t^{i,j} = \text{Covariance of returns on asset } i \text{ with asset } j \text{ over the period } t, t+1$$

[0084] These expected returns and covariance are calculated using the formulas described above. The effective annual expected return for asset j is then calculated as:

$$\mathrm{M}^j = \sum_{t=1}^{T} \omega_t M_t^j$$

Similarly, the effective annual covariance between returns on asset i and returns on asset j are calculated as: (Note, the weights, ω_t, are between zero and one, and sum to one.)

$$\mathrm{Cov}^{i,j} = \sum_{t=1}^{T} \omega_t \mathrm{Cov}_t^{i,j}$$

[0085] In one embodiment, this annualizing technique could be personalized for a given user's situation. For example, the user's horizon could specify T, and their level of current wealth and future contributions could specify the relevant weights. However for

purposes of illustration, the relevant 'effective' moments for optimization and simulation are computed assuming a horizon of 20 years (T=20), and equal weights (i.e. 1/T).

[0086] The techniques described in this section allow for the calculation of the following effective annual moments:

Output parameter name	**Description**	**Units**
M^1	Bills: expected return	Return per year
M^2	Bonds: expected return	Return per year
M^3	Equity: expected return	Return per year
$Cov^{1,1}$	Bills: variance of returns	(Return per year)^2
$Cov^{2,2}$	Bonds: variance of returns	(Return per year)^2
$Cov^{3,3}$	Equity: variance of returns	(Return per year)^2
$Cov^{1,2}$	Bills and Bonds: covariance	(Return per year)^2
$Cov^{1,3}$	Bills and Equity: covariance	(Return per year)^2
$Cov^{2,3}$	Bonds and Equity: covariance	(Return per year)^2

[0087] At step 520, a process for increasing diversification is performed, which is described further below.

[0088] At step 530, a recommended portfolio is output.

DIVERSIFICATION PROCESSING

[0089] Figure 6 is a flow diagram illustrating diversification processing according to one embodiment of the present invention. Conceptually, the diversification processing generally breaks down into an initialization stage, a diversification stage, and an output stage. In the embodiment depicted, the initialization stage is represented by step 622, the diversification stage includes steps 624, 626, and 628, and the output stage is represented by step 629. Briefly, after initializing the candidate portfolio, the diversification stage performs an efficient search of an error space for a more diversified portfolio that can be implemented without exceeding a predetermined diversity budget. The error space is an area proximate to or surrounding the initial candidate portfolio and having boundaries defined in terms of expected return, risk, and/or utility, for example.

[0090] At step 622, the candidate portfolio is initialized to the efficient portfolio that was identified in step 510.

[0091] At step 624, a portfolio that is more diversified than the current candidate portfolio is generated. Various approaches for intelligently identifying a more diverse portfolio than the candidate portfolio are described below.

[0092] At step 626, it is determined whether the cost of implementing the more diversified portfolio is within the diversity budget. If so, then processing continues with step 628; otherwise processing continues with step 629.

[0093] At step 628, the candidate portfolio is updated with the more diversified portfolio and processing continues with step 624. In this manner, the most diversified portfolio within the cost constraints defined by the diversity budget may be identified.

[0094] At step 629, the current candidate portfolio is output as the recommended portfolio.

[0095] Ultimately, since there might be an extremely large number of alternative portfolios of financial products to evaluate, one goal of diversification processing (step 520) is to limit the diversification problem in an intelligent manner. Cost was illustrated above as an exemplary boundary that may act as a stopping condition for diversification processing. As will be explained with reference to Figure 7, various other conditions may be used to terminate the diversification processing. Figure 7 is a flow diagram illustrating diversification processing according to another embodiment of the present invention.

[0096] At step 722, the candidate portfolio is initialized to the efficient portfolio that was identified in step 510.

[0097] At step 724, a portfolio that is more diversified than the current candidate portfolio is generated.

[0098] At step 726, the prior candidate portfolio is set to the current candidate portfolio and the current candidate portfolio is set to the more diversified portfolio and processing continues with step 728. In this manner, depending upon the stopping condition either the portfolio evaluated by the current or prior iteration may be returned as the recommended portfolio depending upon the stopping conditions.

[0099] At step 728, it is determined whether of not one or more stopping conditions has been achieved. If not, then processing continues with step 724; otherwise processing continues with step 729. According to one embodiment, one or more of the following stopping conditions may be used to terminate the diversification processing:

(1) the cost exceeds the diversity budget;

(2) maintaining one or more certain desirable characteristics of the initial candidate portfolio constant is no longer feasible;

(3) the maximum exposure is less than a predetermined minimum exposure threshold;

(4) exposure to a predetermined minimum or maximum number of financial products has been achieved;

(5) a predetermined minimum or maximum number of diversification iterations has been performed; and

(6) a predetermined minimum or maximum number of alternate portfolios has been considered.

[0100] At step 729, either the current candidate portfolio or the prior candidate portfolio is output as the recommended portfolio depending upon the stopping conditions. For example, if the diversity budget has been exceeded by the current candidate portfolio, then the recommended portfolio is set to the last candidate that remained within the diversity budget (e.g., the prior candidate portfolio, in this example). However, if a stopping condition other than diversity budget caused the processing to terminate, then the recommended portfolio may be set to the current candidate portfolio. For example, if the condition causing the diversity processing to terminate was the number of iterations, then the recommended portfolio is set to the current portfolio.

GENERATION OF A MORE DIVERSE PORTFOLIO

[0101] In addition to defining boundaries of the diversification problem in terms of various combinations of stopping conditions, another goal of diversification processing (step 520) is to efficiently search the bounded area (e.g., the error space). Figure 8 is a flow diagram illustrating the generation of a more diverse portfolio (e.g., steps 624 and 724) according to one embodiment of the present invention. According to the embodiment depicted, diversification is achieved by evaluating additional alternative optimal portfolios, using Equation #1 and #2, for example, under various constraints. At step 810, a maximum exposure is selected. The maximum exposure (e.g., UB from above) defines the maximum percentage of the portfolio's value that may be held in any particular financial product for a particular diversification iteration. Importantly, any of a number of approaches may be employed to select the maximum exposure values for iterations of the diversification processing. In one embodiment, the relationship between cost and maximum exposures is assumed to be monotonic. For example, it may be assumed the cost of implementing an efficient portfolio constrained to a maximum exposure of 80% is greater than the cost of implementing an efficient portfolio constrained to a maximum exposure of 90%. In this manner, a search approach that iteratively lowers the ceiling (as defined by the maximum exposure) to search for a more diverse portfolio may stop once a candidate portfolio exceeds the diversity budget. Similarly, a binary search algorithm may be employed that makes use of the monotonic relationship to select the maximum exposure for the current iteration.

[0102] At step 820, optimization processing is performed subject to one or more diversity constraints including the maximum exposure for the current iteration. For example, according to one embodiment, risk is held constant while the maximum exposure constraint is applied. Subsequently, at step 830, one or more characteristics (e.g., expected return, risk, and utility) of the resulting more diversified portfolio are

compared to corresponding characteristics of the initially identified optimal portfolio to measure the cost associated with the current level of diversification.

[0103] Having described various approaches to diversification processing, exemplary iterations are now illustrated with reference to Figures 9A-9C. Figure 9A illustrates an initially identified optimal portfolio 950. Figure 9B illustrates the effect of a maximum exposure constraint on the portfolio of Figure 9A; and Figure 9C illustrates a diversified portfolio after one or more stopping conditions have been achieved.

[0104] In portfolio 950, financial product 910 represents approximately 90% of the portfolio's total value and financial product 920 represents the remaining 10%. According to this example, in a subsequent iteration illustrated by Figure 9B, a maximum exposure constraint 941 of 75% is imposed upon the optimization process to arrive at a more diverse portfolio 951. The cost of implementing portfolio 951 as opposed to portfolio 950 is determined to be within the allocated diversity budget; therefore, another iteration may be performed. Figure 9C represents a more diverse portfolio 952 that results from an even more biting maximum exposure constraint 942. However, the cost, in terms of expected return, risk, and/or utility, of implementing portfolio 952 rather than portfolio 950 is greater than the diversity budget. Therefore, in this example, the recommended portfolio would be portfolio 951 (the most diverse candidate portfolio that stayed within the diversity budget).

[0105] Figure 10 conceptually illustrates an approach for quickly finding a diversified portfolio employing a binary search approach according to one embodiment of the present invention. A maximum exposure 1010 for the first iteration is selected. In this example, the maximum exposure 1010 for the first iteration is 55% (approximately half way between 100% and a floor 1040 of 10%). If the diversity budget is exceeded in the first iteration, then in the next iteration the maximum exposure value is selected to be between 100% and 55% where the cost is known to be lower. In the example of Figure

10, the cost of implementing the candidate portfolio identified by the first iteration is less than the diversity budget; therefore, the maximum exposure value for the second iteration 1020 is selected to be approximately half way between the current exposure and the floor 1040. Subsequent iterations continue in this manner by recursively splitting a remaining portion of the maximum exposure range known to meet the budget constraint until one or more stopping conditions are achieved.

ALTERNATIVE EMBODIMENTS

[0106] Many alternative embodiments are contemplated by the inventors of the present invention. In the foregoing, the diversification stage searched for a more diverse portfolio. Alternately, a similar approach could be used to search for a less diverse portfolio that did not exceed a diversity budget.

[0107] Additionally, expected return was used as an exemplary measure of the cost of diversification. Importantly, however, it should be understood that the present invention is broadly applicable to portfolio diversification approaches that use other measurements of cost, such as risk and/or utility. For example, the expected return on a portfolio could be held constant, and an efficient search could be performed to find a more diverse portfolio within a certain risk budget. Alternatively, diversity may be increased until a given utility budget is exhausted. The utility budget may be defined based upon a user specific utility function which maps any arbitrary characteristics of the portfolio onto a utility measure of desirability, for example. In other embodiments, the optimization problem can be structured to maximize an arbitrary measure of diversity subject to an arbitrary budget.

[0108] Another alternate approach to diversification is to randomly or systematically search all possible portfolios in the error space for a more diverse portfolio that does not exceed a diversity budget. In one embodiment, the area of all possible portfolios is randomly searched until a specified stopping condition has occurred. Stopping

conditions could include: exposure to a minimum or maximum number of mutual funds has been achieved, a predetermined minimum or maximum number of iterations has been performed, or the search has been performed for a specified period of time. A cost associated with a portfolio is calculated by comparing it to the current optimal portfolio. If a newly generated portfolio is more desirable than the current optimal portfolio (e.g., the newly generated portfolio is more diverse and the cost is within the diversity budget), the current optimal portfolio is replaced with the newly generated portfolio. When a stopping condition has occurred, the most diverse portfolio that does not exceed the diversity budget is output.

[0109] Certain aspects of the invention described herein have equal application to various other optimization problems such as those where the inputs into the optimization process are subject to estimation or other types of errors.

[0110] In the foregoing specification, the invention has been described with reference to specific embodiments thereof. It will, however, be evident that various modifications and changes may be made thereto without departing from the broader spirit and scope of the invention. The specification and drawings are, accordingly, to be regarded in an illustrative rather than a restrictive sense.